

Anti-holomorphic twistor and symplectic structure ^{*†}

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Abstract

It is well known that the twistors, section of twistor space, classify the almost complex structure on even dimensional Riemannian manifold X . We will show that existence of a harmonic and anti-holomorphic twistor is equivalent to having a symplectic structure on X

1 Introduction

Recently, the interest of symplectic manifolds has been growing in a perspective of Mathematical Physics related field, for example, Quantum cohomology theory, Seiberg-Witten theory etc. By definition, a manifold having a non-degenerate closed two form ω is called a symplectic manifold. This category of manifolds was firstly understood as that of Kähler manifolds, which has even odd betti number, for example, later on some mathematician like B. Thurston and R. Gompf constructed examples of symplectic manifolds which cannot have Kähler structure. Moreover R. Gompf [Go] find a systematic way of constructing symplectic manifolds and show that every finitely presented group can be realized as a fundamental group of a symplectic 4-manifold. It reveals that the symplectic category is much more bigger than Kähler one and expected to be characterized as cohomology condition of given manifold such as $a \in H^2(X, \mathbf{R})$ and $0 \neq a \cup \cdots \cup a \in H^{2m}(X, \mathbf{R})$. This expectation has been broken in advent of Seiberg-Witten theory for the 4-dimensional topology. It has been known that every symplectic 4-manifold has non-zero Seiberg-Witten invariants [T1, T2], which indicates that condition of having symplectic structure on 4-manifolds is quite subtle. Taking closer look at the Taubes's paper [T1], we can find that he was making use of the characterization of symplectic form, which is there are canonical $Spin^c$ structure associated almost complex structure J and naturally induced a nowhere vanishing positive spinor u which is harmonic, i.e., $\not{D}u = 0$. In this paper, we are going to show that this characterization is equivalent to the existence of the symplectic form on a given manifold. First of all, notice that symplectic form ω on a given manifold realized as an imaginary part of Hermitian metric for some almost complex structure J on TX . Hence the existence of almost complex structure

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is a necessary condition for that of symplectic structure. Given a Riemannian even dimensional manifold, the orthogonal almost complex structure is equivalent to a section of the twistor space which is a canonical fiber bundle of $SO(2m)/U(m)$. We will discuss on this in the section 2. After choosing a twistor u , equivalently having an almost complex structure J , there is the naturally associated $Spin_{2m}$ equivariant Hermitian metric on (TX, J) and a canonical $Spin^C$ representation. The imaginary part of the Hermitian metric ω is our candidate for the symplectic form. It is easily derived that the condition for $d\omega = 0$ is equivalent to the section u is anti-holomorphic and harmonic ($\not{D}u = 0$), where u can be understood as a positive spinor of the canonical positive spinor bundle. To prove this theorem is main purpose of this paper. It also gives a simple characterization of symplectic structure on smooth 4-manifolds, which is the same as the Taubes' analysis of symplectic form. Conclusively, the number of equations for ω being a symplectic form is $m(m-1)/2 + m(m-1)(m-2)/6$ which is bigger than that for integrability condition which is $m(m-1)/2$. That is a little bit odd because the space of symplectic form (deformation space; *it is open in the space of two form* $\Omega^2(X)$) is rather larger than the integrable complex structure which is finite dimension. On the other hand, since the deformation space is a kind of big, there are a lots of such an anti-holomorphic and harmonic twistor for some Riemannian metric on TX once X has a symplectic structure. It gives rise a question whether the condition we have found is "generic", which means in symplectic manifold, the generic Riemannian metric can be induced by a symplectic form ω and an almost-complex structure J associated to it.

2 Pure Spinor and twistor

Fix \mathbf{R}^n be the standard inner product (\langle, \rangle) real vector space and extend this metric \mathbf{C} -linearly to $\mathbf{C}^n = \mathbf{R}^n \otimes \mathbf{C}$. Let $\mathbf{Cl}_n = Cl_n \otimes \mathbf{C}$ be the associated complexified Clifford algebra. Let $\mathcal{S}_{\mathbf{C}}$ be the fundamental \mathbf{Cl}_n -module which defines the irreducible complex spinor space. For each spinor $\sigma \in \mathcal{S}_{\mathbf{C}}$, we can consider the \mathbf{C} -linear map

$$j_\sigma : \mathbf{C}^n \rightarrow \mathcal{S}_{\mathbf{C}} \quad \text{given by } j_\sigma(v) \equiv v \cdot \sigma$$

Generically, this map is injective. However, there are interesting spinors for which $\dim(\ker j_\sigma) > 0$.

Definition 2.1 *A complex subspace $V \subset \mathbf{C}^n$ is said to be **isotropic** (with respect to the bilinear form $\langle \cdot, \cdot \rangle$) if $\langle v, w \rangle = 0$ for all $v, w \in V$.*

We define a hermitian inner product (\cdot, \cdot) on \mathbf{C}^n by setting $(v, w) = \langle v, \bar{w} \rangle$. Clearly, if $V \subset \mathbf{C}^n$ is an isotropic subspace, then $V \perp \bar{V}$ in this hermitian inner product. In particular, therefore, we have

$$2 \dim_{\mathbf{C}} V \leq n.$$

Definition 2.2 *A spinor σ is **pure** if $\ker j_\sigma$ is a maximal isotropic subspace, i.e., if $\dim(\ker j_\sigma) = \lfloor n/2 \rfloor$.*

Denote by $P\mathcal{S}$ the subset of pure spinors in \mathcal{S}_C , and denote by \mathcal{I}_n the set of maximal isotropic subspaces of \mathbf{C}^n . Both $P\mathcal{S}$ and \mathcal{I}_n are naturally acted upon by the group Pin_n , and the assignment $\sigma \mapsto \ker j_\sigma$ gives a Pin_n -equivariant map

$$K : P\mathcal{S} \longmapsto \mathcal{I}_n.$$

From this point on we shall assume that $n = 2m$ is an even integer, and furthermore that \mathbf{R}^{2m} is oriented.

Definition 2.3 *An orthogonal almost complex structure on \mathbf{R}^{2m} is an orthogonal transformation $J : \mathbf{R}^{2m} \rightarrow \mathbf{R}^{2m}$ which satisfies $J^2 = -Id$. For any such J , an associated **unitary basis** of \mathbf{R}^{2m} is an ordered orthonormal basis of the form $\{e_1, Je_1, \dots, e_m, Je_m\}$. Any two unitary bases for a given J determine the same orientation. This is called the **canonical orientation** associated J .*

Let \mathcal{C}_m denote the set of all orthogonal almost complex structures on \mathbf{R}^{2m} . It is easy to see that \mathcal{C}_m is a homogeneous space for the group O_{2m} . It falls into two connected components \mathcal{C}_m^+ and \mathcal{C}_m^- where $\mathcal{C}_m^+ \cong SO_{2m}/U_m$ consists of those almost complex structures whose canonical is **positive** (i.e. agrees with given one on \mathbf{R}^{2m}). Associated to any $J \in \mathcal{C}_m$ there is a decomposition

$$\mathbf{C}^{2m} = V(J) \oplus \overline{V(J)}, \text{ where}$$

$$V(J) \equiv \{v \in \mathbf{C}^{2m} : Jv = -iv\} = \{v_0 + iJv_0 : v_0 \in \mathbf{R}^{2m}\}$$

There is an O_{2m} -equivalent bijection

$$\mathcal{C}_m \xrightarrow{V} \mathcal{I}_{2m}$$

which associates to J the isotropic subspace $V(J)$. Let \mathcal{I}_{2m}^+ denote the component corresponding to \mathcal{C}_m^+ . Using the complex volume element $\omega_{\mathbf{C}} = i^m e_1 \cdots e_{2m}$, we have a decomposition $\mathcal{S}_{\mathbf{C}} = \mathcal{S}_{\mathbf{C}}^+ \oplus \mathcal{S}_{\mathbf{C}}^-$ into $+1$ and -1 eigenspace respectively. Easy calculation gives a decomposition $P\mathcal{S} = P\mathcal{S}^+ \amalg P\mathcal{S}^-$ of the pure spinor space into positive and negative types. Let $\mathbf{P}(P\mathcal{S}^+)$ denote the projectivization of the pure spinor space, i.e., $\mathbf{P}(P\mathcal{S}^+) = P\mathcal{S}^+ / \sim$ where we say that $\sigma \sim \sigma'$ if $\sigma = t\sigma'$ for some $t \in \mathbf{C}$. Each of the space $\mathbf{P}(P\mathcal{S}^\pm)$, \mathcal{C}_m^\pm and \mathcal{I}_{2m}^\pm are acted upon by $Spin_{2m}$, in fact by SO_{2m} .

Proposition 2.4 *The maps $\sigma \mapsto K(\sigma)$ and $J \mapsto V(J)$ induce SO_{2m} -equivariant diffeomorphisms*

$$\mathbf{P}(P\mathcal{S}^+) \xrightarrow{K} \mathcal{I}_{2m}^+ \xrightarrow{V} \mathcal{C}_m^+ \quad \text{and} \quad \mathbf{P}(P\mathcal{S}^-) \xrightarrow{K} \mathcal{I}_{2m}^- \xrightarrow{V} \mathcal{C}_m^-$$

We refer to the original book [LM] for details.

For the sake of further discussion, we will fix $V \in \mathcal{I}_{2m}^+$ and let $J \in \mathcal{C}_m^+$ be the associated complex structure. Choose a unitary basis $\{e_1, Je_1, \dots, e_m, Je_m\}$ of \mathbf{R}^{2m} and set

$$\varepsilon_j = \frac{1}{\sqrt{2}}(e_j - iJe_j) \quad \bar{\varepsilon}_j = \frac{1}{\sqrt{2}}(e_j + iJe_j).$$

Define

$$\omega_j = -\varepsilon_j \bar{\varepsilon}_j \quad \bar{\omega}_j = -\bar{\varepsilon}_j \varepsilon_j \quad (1)$$

Let W be a linear subspace invariant under multiplication by e_j and Je_j . Then there is a hermitian orthogonal direct sum decomposition

$$W = W_j \oplus W'_j$$

where

$$W_j = \bar{\omega}_j \cdot W = \ker(\mu_{\bar{\varepsilon}_j}|_W) \quad \text{and} \quad W'_j = \omega_j \cdot W = \ker(\mu_{\varepsilon_j}|_W)$$

and where $\mu_{\varepsilon_j} : W \rightarrow W$ is defined by $\mu_{\varepsilon_j}(w) = \varepsilon_j \cdot w$. By direct inductive calculation, we can construct

$$\mathcal{S}_m = \ker(\mu_{\bar{\varepsilon}_1}) \cap \cdots \cap \ker(\mu_{\bar{\varepsilon}_m}) \quad \dim_{\mathbb{C}} \mathcal{S}_m = 1$$

The complex volume form $\omega_{\mathbb{C}} = i^m e_1 J e_1 \cdots e_m J e_m$ has the value $+1$ on \mathcal{S}_m because $\bar{\varepsilon}_j \sigma = 0 \Rightarrow -ie_j J e_j \sigma = \sigma$. Therefore, $\mathcal{S}_m \subset \mathcal{S}_{\mathbb{C}}^+$.

We clearly have that $V(J) = \ker j_\sigma$ for $\sigma \in \mathcal{S}_m$. Hence \mathcal{S}_m is independent of the choice of unitary basis and the map $V \mapsto [\mathcal{S}_m]$ gives the desired map K^{-1} for the above proposition.

Definition 2.5 *The bundle $\tau(X) \cong \mathbf{P}(\mathcal{PS}^+)$ is called the **twistor space** of X .*

Note that $\mathbf{P}(\mathcal{PS}^+)$ is an SO_{2m} -bundle and is globally defined whether or not X is a spin manifold.

The total space of $\tau(X)$ carries a canonical almost complex structure defined by using the canonical decomposition of tangent space of $\tau(X)$, which is induced by the Riemannian connection of X .

$$T(\tau(X)) = \mathcal{V} \oplus \mathcal{H}$$

where \mathcal{H} is a field of horizontal planes and \mathcal{V} is the field of tangent planes to the fibers. As noted, \mathcal{V} has an almost complex structure integrable on the fibers since the fiber is naturally homogeneous complex manifold ($\cong SO_{2m}/U(m)$). The bundle \mathcal{H} has a “tautological” almost complex structure defined, via the identification $\pi_* : \mathcal{H}_J \rightarrow TX$, to be the structure J itself.

The question of integrability of J already accomplished by M. Michelsohn.

Theorem 2.6 [LM, M] *Let X be an oriented (even-dimensional) riemannian manifold with an almost complex structure determined by a projective spinor field $u \in \Gamma(\tau(X))$. Then this almost complex structure is integrable if and only if u is holomorphic.*

This will be proved in Remark 3.3. As mentioned above, $\tau(X)$ carries a canonical almost complex structure. Now a C^1 -map between almost complex manifolds $f : (X, J_X) \rightarrow (Y, J_Y)$ will be called **holomorphic** (resp. **anti-holomorphic**) if its differential f_* is everywhere J -linear (resp. anti- J -linear) i.e., if $f_* \circ J_X = \pm J_Y \circ f_*$ respectively.

Remark 2.7 More succinctly one could say that *cross-section of $\tau(X)$ induce almost complex structure, and holomorphic cross-section induce the integrable ones* However, the condition that a cross-section u be holomorphic is *not* linear since the complex structure on X depends itself on u .

We will prove that the complimentary condition for the holomorphicity, which is anti-holomorphic and harmonic is equivalent to that u induce a symplectic structure on X

Definition 2.8 $\omega \in \Omega^2(X)$ is a symplectic form if it is nondegenerate closed form. Moreover, (X, ω) is called a symplectic structure on X .

Given a twistor $u \in \mathbf{P}(\mathcal{PS})$, there is naturally associated nondegenerate differential 2-form. It is induced by the hermitian metric with respect to the almost complex structure J and Riemannian metric g on TX i.e.,

$$\omega(v, w) \equiv g(Jv, w)$$

where J is the almost complex structure corresponding to $s \in \mathbf{P}(\mathcal{PS})$. Moreover it can be written as in terms of unitary basis, in other words, $\omega = \sum_{i=1}^m e_i^* \wedge (Je_i)^*$ where $e^* \in T^*X$ such that $e^*(v) = g(e, v) \in \mathbf{R}$. Recall that $\omega_j = -\varepsilon_j \bar{\varepsilon}_j$ for complex unitary basis $\{\varepsilon_1, \dots, \varepsilon_m, \bar{\varepsilon}_1, \dots, \bar{\varepsilon}_m\}$ of $(TX \otimes \mathbf{C})$. Since $\omega_j = -\varepsilon_j \bar{\varepsilon}_j = 1 - ie_j \cdot Je_j$, $i\omega = m - \sum_j \omega_j$.

$$\begin{aligned} \omega_1 \cdots \omega_m &= \prod (1 - ie_j \cdot Je_j) \\ &= 1 - i \sum e_j \cdot Je_j - \sum_{j \neq k} (e_j \cdot Je_j) \cdot (e_k \cdot Je_k) + \cdots \\ &= 1 - i\omega + (1/2)(-1)^2 i\omega \wedge i\omega + \cdots + (1/m!)(-1)^m i\omega \wedge \cdots \wedge i\omega \\ &= 1 - i\omega + (1/2!)(-i)^2 \omega^2 + \cdots + (1/m!)(-i)^m \omega^m \end{aligned}$$

where $\omega^k = \overbrace{\omega \wedge \cdots \wedge \omega}^{k \text{ times}} \in \Omega^{2k}(X)$.

Remark 2.9 The above equality comes from the identification between TX and TX^* via Riemannian metric. Note that $(1/m!)i^m \omega^m = i^m e_1 \cdot Je_1 \cdots e_m \cdot Je_m = \omega_{\mathbf{C}}$

Note that $*_{\mathbf{C}} \omega^k = k!/(m-k)! \omega^{m-k}$ i.e.,

$$d\omega = 0 \Leftrightarrow d\omega = d^* \omega = 0 \Leftrightarrow \Delta_g(\omega_1 + \cdots + \omega_m) = 0$$

where Δ_g is the Laplacian operator with respect to metric g . Hence we have that ω defines a symplectic form if and only if $\tilde{\omega} = \omega_1 + \cdots + \omega_m$ is harmonic. Our goal is to prove the following theorem.

Theorem 2.10 Let X be an oriented(even-dimensional) riemannian manifold with an almost complex structure determined by a projective spinor field $u \in \Gamma(\tau(X))$. Then this almost complex structure carries symplectic structure if and only if u is harmonic and anti-holomorphic.

The product element , $q = \bar{\omega}_1 \cdots \bar{\omega}_m$ (conjugate of the above product), of the complexified Clifford algebra $\mathbf{Cl}_{2m}(X)$ can be characterized at least locally by an element of $q \in \text{End}(\mathcal{S}^+)$ such that

$$q(\sigma) = \begin{cases} 0 & \text{if } \sigma \in s^\perp \subset \mathcal{S}_{\mathbf{C}} \\ k\sigma & k \in \mathbf{C}^* \text{ and if } [\sigma] = s \end{cases}$$

Note that we have not defined a complex spin representation $\mathcal{S}_{\mathbf{C}}$ globally over X . Without any specification of the complex spinor bundle, the $\bar{\omega}_1 \cdots \bar{\omega}_m$ is well-defined as an element of $\mathbf{Cl}_{2m}(X)$. Using the almost complex structure associated with the twistor u , we can define canonical spin^c structure and canonical complex spin representation. Given the canonical Spin^c representation, the product element $q = 2^m u \otimes u^*$, which is an element of $q \in \text{End}_{\mathbf{C}}(\mathcal{S}^+)$ in a way of that $q(\alpha) = \langle \alpha, u \rangle u$. In the next section, we will prove that $(d\omega)u = 0$ if and only if $\mathcal{D}u = 0$ by using the action of q .

3 Spin^c representation and proof of Theorem 2.10

Since $\text{Spin}_n^c \equiv \text{Spin}_n \times_{\mathbf{Z}_2} U(1)$, we have a short exact sequence

$$0 \longrightarrow \mathbf{Z}_2 \longrightarrow \text{Spin}_n^c \xrightarrow{\xi} SO_n \times U(1) \longrightarrow 1.$$

A principal SO_n -bundle P carries a Spin^c structure if and only if the $w_2(P)$ is the mod 2 reduction of an integral class. Given a twistor $u \in \mathbf{P}(\mathcal{P}^+)$, there is the canonical orthogonal almost complex structure J on TX associated with u . This J defines a canonical Spin^c structure $\det_{\mathbf{C}} TX = K_X^{-1}$ since the first Chern class of K_X^{-1} is an integral lift of the second Stiefel Whitney class, i.e., $c_1(K_X^{-1}) \equiv w_2(X) \text{ mod } 2$. Let $\mathcal{S}_{\mathbf{C}}$ be the associated spinor bundle. Using the complex volume form $\sqrt{-1}^m e_1 \cdot J e_1 \cdots e_m \cdots J e_m$, we have the decomposition of $\mathcal{S}_{\mathbf{C}}$ by the \pm -eigenspace of the complex volume element, where $\mathcal{S}^\pm = (1 \pm \omega_{\mathbf{C}})\mathcal{S}_{\mathbf{C}}$. Set

$$\varepsilon_j = \frac{1}{\sqrt{2}}(e_j - iJ e_j) \quad \bar{\varepsilon}_j = \frac{1}{\sqrt{2}}(e_j + iJ e_j).$$

be an unitary basis for TX as above. Define

$$\mathcal{S}_{\mathbf{C}} \cong \oplus \mathcal{S}_{i_1, \dots, i_m} \cong \oplus \ker(\mu_{\varepsilon_{i_1}}) \cap \cdots \cap \ker(\mu_{\varepsilon_{i_m}})$$

where $\mu_{\varepsilon_{i_k}} = \begin{cases} \mu_{\varepsilon_k} & i_k = k \\ \mu_{\bar{\varepsilon}_k} & i_k = \bar{k} \end{cases}$. Let $\sigma = \{i_1, \dots, i_m\}$ be the complex index used as above, define $|\sigma|$ be the number of elements of the subset $\{i_k = k\}$. Then we have

$$\mathcal{S}_{\mathbf{C}}^+ \cong \bigoplus_{|\sigma|=2i} \mathcal{S}_{i_1 \dots i_m} \quad \mathcal{S}_{\mathbf{C}}^- \cong \bigoplus_{|\sigma|=2i-1} \mathcal{S}_{i_1 \dots i_m}$$

Especially, the twistor u is contained in $\mathcal{S}_{1, \dots, \bar{m}}$ which is characterized as $\bar{\varepsilon}_j \cdot u = 0$ for all j . We can express the Dirac operator in terms of the unitary basis, which

follows that

$$\begin{aligned}
\mathbf{D} &= e_j \cdot \nabla_{e_j} + J e_j \cdot \nabla_{J e_j} \\
&= \frac{1}{2}(\varepsilon_j + \bar{\varepsilon}_j) \cdot \nabla_{\varepsilon_j + \bar{\varepsilon}_j} - \frac{1}{2}(\varepsilon_j - \bar{\varepsilon}_j) \cdot \nabla_{\varepsilon_j - \bar{\varepsilon}_j} \\
&= \bar{\varepsilon}_j \cdot \nabla_{\varepsilon_j} + \varepsilon_j \cdot \nabla_{\bar{\varepsilon}_j}
\end{aligned}$$

Remark 3.1 *Note that the covariant derivative ∇ is Spin^c connection which is induced from both the Levi-Civita connection and the $U(1)$ connection on K_X^{-1} . It should be well-noticed that our theorem is nothing to do with a $U(1)$ connection. Even though the condition we have imposed is related to simply “local” question, the spin^c structure enable us to work with globally. Furthermore, the following argument we will present below works finely without any spin^c structure.*

To define a Dirac operator on the spinors, we should specify a $U(1)$ connection on K_X^{-1} . There is a canonical $U(1)$ connection unique up to gauge transformation A_0 such that $\langle \nabla u, u \rangle = 0$. We will abuse the notation \mathbf{D} for the Dirac operator, \mathbf{D}_{A_0} , which is induced by the Levi-Civita connection and the canonical $U(1)$ connection A_0 . Our index notation convention indicates that $\nabla_{\tilde{e}_j} \tilde{e}_k = \sum_l \tilde{\omega}_k^l(\tilde{e}_j) e_l$ and $\Gamma_{j,k}^l = \tilde{\omega}_k^l(\tilde{e}_j)$ is the Christoffel symbol. Let $e_j = \tilde{e}_{2j-1}$, $J e_j = \tilde{e}_{2j}$ then $\varepsilon_j = \frac{1}{\sqrt{2}}(\tilde{e}_{2j-1} - i \tilde{e}_{2j})$ where $i = \sqrt{-1}$. Then we have

$$\begin{aligned}
\nabla_{\varepsilon_j} \bar{\varepsilon}_k &= a_{j,k}^l \varepsilon_l + c_{j,k}^l \bar{\varepsilon}_l, & \nabla_{\bar{\varepsilon}_j} \varepsilon_k &= \bar{a}_{j,k}^l \bar{\varepsilon}_l + \bar{c}_{j,k}^l \varepsilon_l \\
\nabla_{\bar{\varepsilon}_j} \bar{\varepsilon}_k &= b_{j,k}^l \varepsilon_l + d_{j,k}^l \bar{\varepsilon}_l, & \nabla_{\varepsilon_j} \varepsilon_k &= \bar{b}_{j,k}^l \bar{\varepsilon}_l + \bar{d}_{j,k}^l \varepsilon_l
\end{aligned}$$

Since the Levi-Civita connection is naturally compatible with the Hermitian metric on $TX \otimes \mathbf{C}$, we have

$$a_{j,k}^l = \langle \nabla_{\varepsilon_j} \bar{\varepsilon}_k, \varepsilon_l \rangle = - \langle \bar{\varepsilon}_k, \nabla_{\bar{\varepsilon}_j} \varepsilon_l \rangle = -a_{j,l}^k$$

By the same manner, we have

$$b_{j,k}^l = -b_{j,l}^k \quad \text{and} \quad c_{j,k}^l = -\bar{d}_{j,l}^k$$

Lemma 3.2 *Let u be a section of twistor space and J be the associated orthogonal almost complex structure. Then u is anti-holomorphic if and only if $a_{j,k}^l = 0$ for all j, k, l and u is holomorphic section if and only $b_{j,k}^l = 0$ for all j, k, l .*

First of all, we have to find the covariant derivative of u which is

$$\nabla u = \frac{1}{2} \sum_{k < l} \tilde{\omega}_k^l \otimes \tilde{e}_l \tilde{e}_k \cdot u$$

where $\tilde{\omega}$ is the $so(2m)$ connection 1-form (Levi-Civita connection with respect to g) associated with orthonormal basis $\{\tilde{e}_1, \dots, \tilde{e}_{2m}\}$. Let

$$\nabla_{\varepsilon_j} u \equiv \frac{1}{2} \sum_{k < l} \tilde{a}_{j,k}^l \otimes \varepsilon_l \cdot \varepsilon_k \cdot u \quad \text{mod } \langle u \rangle$$

The coefficient $\tilde{a}_{j,k}^l$ can be derived as follows,

$$\bar{\varepsilon}_t \cdot u = 0 \quad \text{for all } t$$

By taking covariant derivative ∇_{ε_j} , we have

$$(\nabla_{\varepsilon_j} \bar{\varepsilon}_t) \cdot u + \bar{\varepsilon}_t \cdot \nabla_{\varepsilon_j} u = 0$$

Hence

$$\begin{aligned} (\nabla_{\varepsilon_j} \bar{\varepsilon}_t) \cdot u &= -\bar{\varepsilon}_t \cdot \sum_{k < l} \frac{1}{2} \tilde{a}_{j,k}^l \varepsilon_l \varepsilon_k \cdot u \\ &= -\sum_{k < l} \frac{1}{2} \tilde{a}_{j,k}^l \bar{\varepsilon}_t \varepsilon_l \varepsilon_k \cdot u \\ &= \begin{cases} \tilde{a}_{j,k}^l \varepsilon_l \cdot u & \text{for } k = t < l \\ -\tilde{a}_{j,k}^t \varepsilon_k \cdot u & \text{for } l = t > k \end{cases} \end{aligned}$$

Since $\bar{\omega}_j = -\bar{\varepsilon}_j \cdot \varepsilon_j \cdot u = 2u$. Therefore

$$\langle \nabla_{\varepsilon_j} \bar{\varepsilon}_t, \varepsilon_s \rangle = a_{j,t}^s = \tilde{a}_{j,t}^s$$

We get $\tilde{a}_{j,k}^l = \langle \nabla_{\varepsilon_j} \varepsilon_k, \varepsilon_l \rangle$. By the analogous method, we can get

$$\nabla_{\bar{\varepsilon}_j} u \equiv \frac{1}{2} \sum_{k < l} b_{j,k}^l \otimes \varepsilon_l \cdot \varepsilon_k \cdot u \quad \text{mod } \langle u \rangle.$$

With this understood, it can be rephrased that u is anti-holomorphic $\Leftrightarrow \bar{\varepsilon}_t \cdot \nabla_{\varepsilon_j} u = 0 \Leftrightarrow a_{j,k}^l = 0 \Leftrightarrow \langle \nabla_{\varepsilon_j} \bar{\varepsilon}_k, \varepsilon_l \rangle = 0$ for all j, k, l . Also u is holomorphic $\Leftrightarrow \bar{\varepsilon}_t \cdot \nabla_{\bar{\varepsilon}_j} u = 0 \Leftrightarrow a_{j,k}^l = 0 \Leftrightarrow \langle \nabla_{\varepsilon_j} \bar{\varepsilon}_k, \varepsilon_l \rangle = 0$ for all j, k, l .

Remark 3.3 From the torsion free condition of Levi-Civita connection, we have

$$b_{j,k}^l - b_{k,j}^l = \langle \nabla_{\bar{\varepsilon}_j} \bar{\varepsilon}_k - \nabla_{\bar{\varepsilon}_k} \bar{\varepsilon}_j, \varepsilon_l \rangle = \langle [\bar{\varepsilon}_j, \bar{\varepsilon}_k], \varepsilon_l \rangle.$$

Since the anti-commutativity between upper index and right lower index $b_{j,k}^l = -b_{j,l}^k$, we can get an equivalent condition which says that $b_{j,k}^l - b_{k,j}^l = 0$ if and only if $b_{j,k}^l = 0$. Hence it is easy to prove the Theorem 2.6 from the above equation.

We want to find an equivalent condition for the harmonic two form ω i.e. $\mathcal{D}\omega = (d + d^*)\omega = 0$, where d^* is the formal adjoint of d with respect to g . The following lemma is about it.

Proposition 3.4 Let $\omega = -m + \sum_k \omega_k$ be the purely imaginary part of the Hermitian metric. Then $\mathcal{D}\omega = 0$ if and only if $a_{j,k}^l = 0$ and $b_{j,k}^l + b_{l,j}^k + b_{k,l}^j = 0$ for all j, k, l .

Proof: Since ω is purely imaginary two form, we have

$$\begin{aligned}\mathbb{D}\omega &= \sum_j \varepsilon_j \nabla_{\bar{\varepsilon}_j} \omega + \bar{\varepsilon}_j \nabla_{\varepsilon_j} \omega \\ &= \sum_j \varepsilon_j \nabla_{\bar{\varepsilon}_j} \omega - \overline{\sum_j \varepsilon_j \nabla_{\bar{\varepsilon}_j} \omega} \\ &= 2i \operatorname{Im} \mathbb{D}^{\frac{1}{2}} \omega\end{aligned}$$

It suffices to consider the half part of the Dirac operator, it reads

$$\begin{aligned}-\mathbb{D}^{\frac{1}{2}} \omega &= \sum_j -\varepsilon_j \cdot \nabla_{\bar{\varepsilon}_j} \omega \\ &= \sum_{j,k} (\varepsilon_j \cdot (\nabla_{\bar{\varepsilon}_j} \varepsilon_k) \cdot \bar{\varepsilon}_k + \varepsilon_j \cdot \varepsilon_k \cdot \nabla_{\bar{\varepsilon}_j} \bar{\varepsilon}_k) \\ &= \sum_{j,k,l} (\bar{a}_{j,k}^l \varepsilon_j \bar{\varepsilon}_l \bar{\varepsilon}_k + \bar{c}_{j,k}^l \varepsilon_j \varepsilon_l \bar{\varepsilon}_k + b_{j,k}^l \varepsilon_j \varepsilon_k \varepsilon_l + d_{j,k}^l \varepsilon_j \varepsilon_k \bar{\varepsilon}_k) \\ &= \sum_{j,k,l} (\bar{a}_{j,k}^l \varepsilon_j \bar{\varepsilon}_l \bar{\varepsilon}_k + b_{j,k}^l \varepsilon_j \varepsilon_k \varepsilon_l) + \sum_{j,k,l} (\bar{c}_{j,l}^k + d_{j,k}^l) \varepsilon_j \varepsilon_k \bar{\varepsilon}_l \\ &= \sum_{i,j,l} (\bar{a}_{j,k}^l \varepsilon_j \bar{\varepsilon}_l \bar{\varepsilon}_k + b_{j,k}^l \varepsilon_j \varepsilon_k \varepsilon_l)\end{aligned}$$

Hence $\mathbb{D}\omega = 0$ if and only if $\mathbb{D}^{\frac{1}{2}} \omega = 0 \Leftrightarrow a_{j,k}^l = 0$ for all i, j, k and $\sum_{\sigma} b_{\sigma(j), \sigma(k)}^{\sigma(l)}$ where σ is the permutation of i, j, k . The relation $b_{j,k}^l = -b_{j,l}^k$ completes the proposition.

Proof of Theorem 2.10 Since $\langle \nabla u, u \rangle = 0$, we have

$$\begin{aligned}\mathbb{D}u &= \sum_j (\bar{\varepsilon}_j \nabla_{\varepsilon_j} u + \varepsilon_j \nabla_{\bar{\varepsilon}_j} u) \\ &= \sum_{j,k,l} \left(\frac{1}{4} a_{j,k}^l \bar{\varepsilon}_j \varepsilon_k \varepsilon_l \cdot u + \frac{1}{4} b_{j,k}^l \varepsilon_j \varepsilon_k \varepsilon_l \cdot u \right) \\ &= \sum_{j,k} a_{j,j}^k \varepsilon_k \cdot u + \sum_{j < k < l} \frac{1}{2} (b_{j,k}^l + b_{l,j}^k + b_{k,l}^j) \varepsilon_j \varepsilon_k \varepsilon_l \cdot u\end{aligned}$$

Hence u is anti-holomorphic pure spinor ($\bar{\varepsilon}_t \nabla_{\varepsilon_j} u = 0$ for all t, j) and harmonic ($\mathbb{D}u = 0$) gives an equivalent condition for ω being a symplectic form. Note that given symplectic manifold (X, ω) has such a anti-holomorphic and harmonic twistor u by choosing any almost complex structure which calibrate ω .

Corollary 3.5 $(d\omega) \cdot u = 0$ if and only if u is harmonic, i.e., $\mathbb{D}u = 0$.

Proof: Let $q = \prod_{j=1}^m \bar{\omega}_j = \prod_j (1 + i\varepsilon_j \cdot J\varepsilon_j)$. Using the action q on u , $q \cdot u = 2^m u$, and taking Dirac operator on the both side, we can have

$$\begin{aligned}\mathbb{D}q \cdot u &= (\mathbb{D}q) \cdot u + \sum \tilde{\varepsilon}_j q \cdot \nabla_{\tilde{\varepsilon}_j} u \\ &= \mathbb{D}q \cdot u \Leftrightarrow (\langle \nabla u, u \rangle = 0 \Rightarrow q \cdot \nabla_{\tilde{\varepsilon}_j} u = 0) \\ &= \mathbb{D}u\end{aligned}$$

Thus $\mathbb{D}u = 0$ if and only if $(\mathbb{D}q) \cdot u = 0$. Moreover since $(3i)^m(-1)^{\frac{1}{2}p(p+1)}\varphi\omega_{\mathbf{C}} = *\varphi$ for $\varphi \in \Omega^p(X)$ and $1/k! * \omega^k = 1/(m-k)!\omega^{m-k}$, we have

$$\begin{aligned}\mathbb{D}q &= (d + d^*)q = id\omega + i^2/2!d\omega^2 + \cdots + i^{m-1}/(m-1)!d\omega^{m-1} \\ &\quad - i/(m-1)! * d\omega^{m-1} - i^2/(m-2)! * d\omega^{m-2} - \cdots - i^{m-1} * d\omega \\ &= id\omega + i^2/2!d\omega^2 + \cdots + i^{m-1}/(m-1)!d\omega^{m-1} \\ &\quad i^{4m+1}d\omega \cdot \omega_{\mathbf{C}} + i^{4m+2}/2!d\omega^2 \cdot \omega_{\mathbf{C}} \cdots + i^{m-1}/(m-1)!d\omega^{m-1} \cdot \omega_{\mathbf{C}}\end{aligned}$$

Since $\omega_{\mathbf{C}}u = u$, $\omega \cdot u = -(mi)u$, we have $d\omega^k \cdot u = k(d\omega) \wedge \omega^{k-1}u = k(-mi)^{k-1}d\omega \cdot u$. Thus

$$(\mathbb{D}q) \cdot u = 2i(1 + m + m^2/2! + \cdots + m^{m-2}/(m-2)!)d\omega \cdot u.$$

This completes the proof.

Remark 3.6 In dimension $2m \leq 6$ every non-zero positive (or negative) spinor is pure, i.e., $P\mathbb{S}^{\pm} = \mathbb{S}_{\mathbf{C}}^{\pm} - 0$. This is simply because the group $Spin_{2m}$ acts transitively on the unit sphere in $\mathbb{S}_{\mathbf{C}}^{\pm}$ in these dimensions.

In dimension 4, since $\varphi \in \Omega^3(X, \mathbf{R})$ acts on u injectively, we get $(d\omega) \cdot u = 0$ if and only if $d\omega = 0$. Hence the harmonic spinor u , equivalently anti-holomorphic twistor, gives a sufficient condition to induce a symplectic structure. The next corollary follows from it.

Corollary 3.7 In dimension 4, Let u be a nowhere vanishing section of positive complex spinor bundle. Then $\mathbb{D}u = 0$ and $\langle \nabla u, u \rangle = 0$ then X is symplectic 4-manifold.

Finally, suppose $\nabla u = 0$, then u is then both holomorphic and anti-holomorphic twistor. We have following corollary, which is proposition 9.8 in [LM].

Corollary 3.8 Suppose u is parallel, then (X, g, J) becomes a Kähler manifold.

References

- [D] S.K. Donaldson. The Seiberg-Witten equations and 4-manifold topology. *Bull. Amer. Math. Soc*, 33:45–70, 1996.
- [Go] R. Gompf. A new construction of symplectic manifolds. *Ann. of Math.* 142(1995), 527-595
- [LM] Jr. Lawson, H. Blaine and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.
- [M] John W. Morgan. *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, volume 44 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1996.

- [T1] Clifford Henry Taubes. The Seiberg-Witten invariants and symplectic forms. *Math. Res. Lett.*, 1(6):809–822, 1994.
- [T2] Clifford Henry Taubes. More constraints on symplectic forms from Seiberg-Witten invariants. *Math. Res. Lett.*, 2(1):9–13, 1995.